The empty set

One of the most important sets in mathematics is the empty set, $\varnothing$.

This set contains no elements.

When one defines a set via some characteristic property, it may be the case that there exist no elements with this property. If so, the set is empty.

For instance, the set of real numbers x such that $x^2 + 5 = 0$ is empty.

The set of flamingos teaching at Drexel is empty.

It is sometimes difficult to determine if a given set contains any elements.

Thus, it is not known if the set of digits that occur finitely many times in the decimal expansion of $\pi = 3.1415926535897932...$ is nonempty.

The role of $\varnothing$ among sets is analogous to the role of zero in a number system.

The equalities

$$X \cup \varnothing = X$$
$$X \cap \varnothing = \varnothing$$

hold for any set X. Furthermore,

$$X \cap Y = \varnothing, \qquad \text{if } X \text{ and } Y \text{ are disjoint}$$
$$X \setminus Y = \varnothing, \qquad \text{if } X \text{ is a subset of } Y.$$

Every nonempty set has at least two subsets, $\varnothing$ and itself. The empty set has only one, itself. The empty set is a subset of any other set, but not necessarily an element of it.

The following are true statements:

$$\varnothing \notin \varnothing, \qquad \varnothing \neq \{\varnothing\}, \qquad \varnothing \in \{\varnothing\},$$
$$\varnothing \subset \{\varnothing\}, \qquad \varnothing \notin \{1\}, \qquad \varnothing \subset \{1\},$$
$$\{\varnothing\} \cap \{1\} = \varnothing, \qquad \varnothing \cup \{1\} = \{1\}, \qquad \{\varnothing\} \cup \{1\} = \{\varnothing,\ 1\},$$
$$\{1\} \setminus \{1\} = \varnothing.$$